VISCOUS-FLUID FLOW IN A THIN LAYER ON THE SIDE SURFACE OF A ROTATING BRAKING-UPPER-END CYLINDER

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An analytical solution is obtained to the hydrodynamic problem on circulating viscous-fluid flow in a thin layer on the side surface of a rotating braking-end cylinder. Calculated data are in satisfactory agreement with the experimental findings and with the numerical integration of the equations of motion.

An experimental study has been made of the hydrodynamic characteristics of the viscous-fluid flow in a thin layer on the side surface of a rotating braking-end cylinder [1-4]. A three-dimensional flow structure, in which thin boundary layers usually develop on end surfaces, requires that rather complex computational methods [3, 4] be used in the theoretical analysis. The present work is aimed at constructing an analytical solution to the above-mentioned problem. Having a sufficient accuracy, such a solution must be simple and clear, thus allowing its use in design calculations.

We consider a viscous, incompressible thin-layer fluid flow on the side surface of a rotating cylinder by allowing for the braking-upper-end influence. Such flow conditions are realized when a right circular cylinder of radius R that is partially filled with fluid starts rotating with a large angular velocity ω around its symmetry axis. As a result of centrifugal forces, the fluid rises along the side wall of the cylinder towards its upper end, thus forming a thin layer rotating together with the cylinder (Fig. 1).

If the upper end of the cylinder rotates with an angular velocity less than ω or it is fixed, then, by virtue of the rotating flow drag, in the developed boundary layer there appears an unbalance of the centrifugal force and the pressure gradient that causes secondary radial flow to the axis of the cylinder; with the latter being closed outside the boundary layer, the circulating fluid flow takes place.

We assume that the centrifugal force significantly exceeds the force of gravity. As a result, layer thickness h may be considered to be independent of the z coordinate. In a thin-layer approximation (h << R), the Navier-Stokes equations, which describe the steady fluid flow, in the Cartesian coordinates with the x axis originating on the cylinder surface and directed inward, and with the z axis measured from the surface of the lower end of the cylinder and directed upward are of the form [5]:

$$v_x \frac{\partial v_x}{\partial x} + \frac{v_y^2}{R} + v_z \frac{\partial v_x}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2} \right), \quad (1)$$

$$v_x \frac{\partial v_y}{\partial x} + v_z \frac{\partial v_y}{\partial z} - \frac{v_x v_y}{R} = \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2} \right), \quad (2)$$

$$v_x \frac{\partial v_z}{\partial x} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} \right), \quad (3)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0. \quad (4)$$

First, we consider the problem of the small perturbations introduced in the rotating flow by the upper end of the cylinder. In this case, to a first approximation, one may set

and Eq. (2) which is used to determine v_{y0} takes the form

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$$v_{x0} = 0, \; v_{z0} = 0, \tag{5}$$

Owing to the fluid sticking on solid surfaces and to the lack of torque on the inner layer surface, the boundary conditions for v_{y0} become:

$$\frac{\partial^2 v_{y0}}{\partial x^2} + \frac{\partial^2 v_{y0}}{\partial z^2} = 0. \tag{6}$$

By solving Eq. (5) and using the boundary conditions (6), we arrive at:

$$v_{y0}(x, L) = 0, \; v_{y0}(0, z) = \omega R, \; v_{y0}(x, 0) = \omega R, \; \frac{\partial v_{y0}}{\partial x}(h, z) = 0. \tag{7}$$

where $\beta = (\pi/h)/(n + 1/2)$.

Now, Eq. (7) can be used to estimate secondary circulating flows. For this purpose, a pressure p(x, z) can be given as a sum of two components

$$v_{y0} = \omega R - \sum_{n=0}^{\infty} \frac{2\omega R \,\mathrm{sh}(\beta_n z) \sin(\beta_n x)}{h\beta_n \,\mathrm{sh}(\beta_n L)}, \tag{8}$$

where $p_0(x)$ obeys the equation $-\omega^2 R = (1/\rho)/(\partial p_0/\partial x)$.

Neglecting the inertia terms in Eq. (1) and Eq. (3), using the condition (8), and including the stream function ψ yields

$$p(x, z) = p_0(x) + p_1(x, z),$$
$$-2\frac{\partial v'_{y0}}{\partial z} = \frac{\nu}{\omega}\frac{\partial^4 \psi}{\partial x^4}, \tag{9}$$

where $v'_{y0} = v_{y0} - \omega R$. In deriving Eq. (9), we used the assumption ($h << L$) that the stream-function change along the z axis is much less than its change along the x axis. The boundary conditions for the stream function assume the form:

$$\psi(0, z) = 0, \; \psi(h, z) = 0, \; \frac{\partial \psi}{\partial x}(0, z) = 0, \; \frac{\partial^2 \psi}{\partial x^2}(h, z) = 0. \tag{10}$$

Integrating (9) and using Eq. (10) yields:

$$\psi = \frac{4\omega^2 R}{\nu h} \sum_{n=0}^{\infty} \frac{\mathrm{ch}\,\beta_n z}{\mathrm{sh}\,\beta_n L} \left\{ \frac{\sin \beta_n x}{\beta_n^4} + \frac{x^3}{2h^3\beta_n^2}\left[(-1)^n\left(\frac{1}{\beta_n^2} + \frac{h^2}{2}\right) - \frac{h}{\beta_n}\right] - \frac{3x^2}{2h^2\beta_n^2}\left[(-1)^n\left(\frac{1}{\beta_n^2} + \frac{h^2}{6}\right) - \frac{h}{\beta_n}\right] - \frac{x}{\beta_n^3}\right\}, \tag{11}$$

$$v_z = \frac{4\omega^2 R}{\nu h} \sum_{n=0}^{\infty} \frac{\mathrm{ch}\,\beta_n z}{\beta_n^3 \,\mathrm{sh}\,\beta_n L} \left\{ \cos \beta_n x + \frac{3x^2}{2h}\beta_n\left[(-1)^n\left(\frac{1}{2} + \frac{1}{h^2\beta_n^2}\right) - \frac{1}{h\beta_n}\right] - x\beta_n\left[(-1)^n\left(\frac{1}{2} + \frac{3}{h^2\beta_n^2}\right) - \frac{3}{h\beta_n}\right] - 1\right\}. \tag{12}$$

The solution is valid under the condition that

$$\mathrm{Re} = \frac{\omega h^2}{\nu} \ll 1, \tag{13}$$

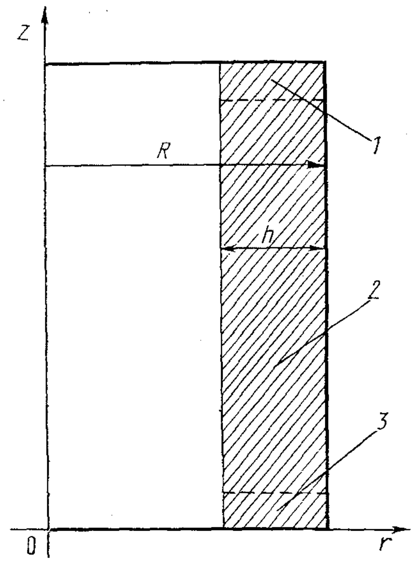

Fig. 1. Lay-out scheme of flow regions: (1), (3), boundary layers on the upper and lower ends of the cylinder, respectively; (2), flow core.

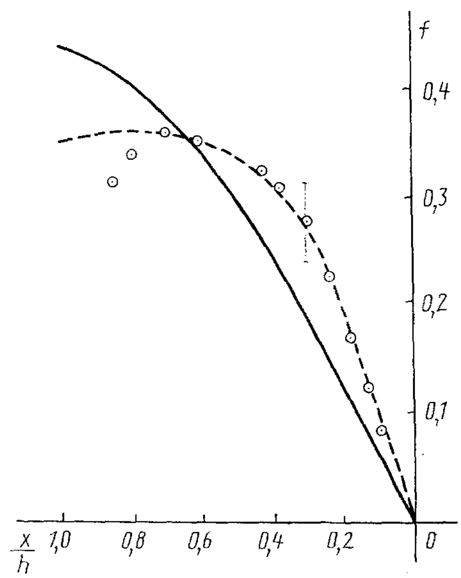

Fig. 2. Dimensionless azimuthal flow velocity profiles in the rotating coordinate system: solid curve, analytical calculation; dashed curve, results of a numerical integration [4]; points, experimental data [4].

which allows the nonlinear inertia terms to be neglected in the equations of motion. This condition is satisfied only when the cylinder rotates relatively slowly. In practice, usually $Re >> 1$ [1, 4]. As a result, boundary layers develop on the end surfaces of the cylinder, and calculations should include the nonlinear inertia terms.

We estimate the azimuthal velocity component by using the system of Navier-Stokes equation, including the inertia terms.

We subdivide the entire flow region into three zones: upper and lower boundary layers on cylinder ends and the flow core. In the flow core, the radial (u_0) and azimuthal (v_0) velocity components are assumed not to depend on the axial z coordinate. A solution for the azimuthal velocity component in the flow core will be sought in the form

$$v_0 = \omega r + v. \tag{14}$$

Using the projection of the equation of motion onto the φ axis in the cylindrical coordinate system and passing to a variable $x = R - r$, we arrive at:

$$-2u_0\omega + u_0 \frac{dv}{dx} = \nu \frac{d^2v}{dx^2}. \tag{15}$$

Upon integrating (15) and using the conditions both for fluid sticking to the side surface of the cylinder ($v(0) = 0$) and for a lack of viscous friction forces at the layer boundary ($(dv/dx)(h) = 0$), we have

$$v_0 = \omega \left[R + x + \frac{2\nu}{u_0} \left(1 - e^{\frac{u_0 x}{\nu}}\right) e^{-\frac{u_0 h}{\nu}} \right]. \tag{16}$$

For the unknown quantity u_0 to be determined, we consider the fluid motion in the boundary layers on the system ends. For this, the analysis to be made uses the integral Karman relations [6]. For the upper boundary layer in a planar geometry approximation, these relations are of the form:

$$-\frac{d}{dx}\left[\int\limits_{L-\delta_0}^{L} v_x^2 dz\right] - \frac{1}{R}\int\limits_{L-\delta_0}^{L} v_y^2\, dz = -\nu \left.\frac{\partial v_x}{\partial z}\right|_{L-\delta_0}^{L} - \frac{v_0^2}{R}\,\delta_0, \tag{17}$$

$$\frac{d}{dx}\left[\int\limits_{L-\delta_0}^{L} v_x v_y dz\right] - v_0 \frac{d}{dx}\left[\int\limits_{L-\delta_0}^{L} v_x dz\right] = \nu \left.\frac{\partial v_y}{\partial z}\right|_{L-\delta_0}^{L}. \tag{18}$$

For the lower-disc boundary layer, similar relations are obtained

$$-\frac{d}{dx}\left[\int\limits_{0}^{\delta_1} (v_x')^2\, dz\right] - \frac{1}{R}\int\limits_{0}^{\delta_1} (v_y')^2\, dz = -\nu \left.\frac{\partial v_x'}{\partial z}\right|_{0}^{\delta_1} - \frac{v_0^2}{R}\,\delta_1, \tag{19}$$

$$\frac{d}{dx}\left[\int\limits_{0}^{\delta_1} v_x' v_y'\, dz\right] - v_0 \frac{d}{dx}\left[\int\limits_{0}^{\delta_1} v_x'\, dz\right] = \nu \left.\frac{\partial v_y'}{\partial z}\right|_{0}^{\delta_1}. \tag{20}$$

Here δ_0 and δ_1 are the thicknesses of the upper and lower boundary layers, respectively.

We assume that the quantities u_0, δ_0, and δ_1 weakly depend on the x coordinate. In fact, this means that in the flow core and in the boundary layers the flow reversal zones both near the side surface of the cylinder and near the external layer boundary are ignored.

From the continuity equation for the radial flow we have

$$(L - \delta_1 - \delta_0)\, u_0 + \int\limits_{0}^{\delta_1} v_x' dz + \int\limits_{L-\delta_0}^{L} v_x dz = 0. \tag{21}$$

Approximately, we represent the velocity profiles in the boundary layers as the following quadratic polynomials:

$$v_x = u_0 \frac{(L-z)}{\delta_0}\left[b + (1-b)\frac{L-z}{\delta_0}\right],$$

$$v_y = \left[2\frac{(L-z)}{\delta_0} - \left(\frac{L-z}{\delta_0}\right)^2\right] v_0(h),$$

$$v_x' = u_0 \frac{z}{\delta_1}\left[a + (1-a)\frac{z}{\delta_1}\right], \tag{22}$$

$$v_y' = \omega R + [v_0(h) - \omega R]\left(2\frac{z}{\delta_1} - \frac{z^2}{\delta_1^2}\right).$$

Substituting Eq. (22) into Eqs. (17)-(21) and taking into account the above assumptions yields a system of 5 algebraic equations for finding the unknown quantities u_0, δ_0, δ_1, a, and b:

$$\frac{7}{15}\frac{v_0^2(h)}{R}\delta_0 = -\frac{2(1-b)u_0}{\delta_0}\nu, \tag{23}$$

$$\frac{u_0\delta_0^2}{2h\nu}\left(\frac{11}{60} + \frac{b}{20}\right) = 1, \tag{24}$$

$$-\frac{\delta_1}{R}\left[\omega^2R^2 + \frac{4}{3}(v_0(h) - \omega R) + \frac{8}{15}(v_0(h) - \omega R)^2\right] = -\frac{\delta_1}{R}v_0^2(h) - \frac{2\nu(1-a)u_0}{\delta_1}, \tag{25}$$

$$-\frac{u_0\delta_1}{h\nu}\left(\frac{a}{20} + \frac{1}{30}\right) = 2, \tag{26}$$

$$Lu_0 + u_0\delta_1\left(\frac{a}{6} + \frac{1}{3}\right) + u_0\delta_0\left(\frac{b}{6} + \frac{1}{3}\right) = 0. \tag{27}$$

Under the conditions $|a| >> 1$ and $|b| >> 1$, from Eqs. (23)-(27) we have

$$\delta_0 = \left(\frac{1200h}{7RV^2}\right)^{1/4}\left(\frac{\nu}{\omega}\right)^{1/2},$$

$$\delta_1 = \left(\frac{80h}{RD}\right)^{1/4}\left(\frac{\nu}{\omega}\right)^{1/2},$$

$$b = \frac{120VR\nu}{7u_0\delta_0^2}, \quad a = \frac{120(V-1)R\nu}{7u_0\delta_1^2}, \quad V = \frac{v_0(h)}{\omega R},$$

$$D = 1 - V^2 + \frac{4}{3}(V-1) + \frac{8}{15}(V-1)^2.$$

The unknown dimensionless radial velocity $U = u_0h/\nu$ is found by solving the following transcendental equation

$$\frac{V-1}{\Delta_1} + \frac{V}{\Delta_0} = -\frac{7U}{20A},$$

where

$$A = \frac{R}{L}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}h, \quad \Delta_0 = \delta_0\left(\frac{\omega}{\nu}\right)^{1/2}, \quad \Delta_1 = \delta_1\left(\frac{\omega}{\nu}\right)^{1/2}.$$

Calculating the quantity U as applied to the parameters of a device described elsewhere [4] (R = 0.095 m, h = 0.019 m, L = 0.19 m, ω = 100 1/sec) yields U $\simeq$ −0.89. Figure 2 shows azimuthal velocity component distributions with respect to the rotating coordinate system (f(x/h) = 1 − V) in the flow core. A satisfactory agreement between the constructed analytical solution, the experimental data, and the numerical solution to the Navier-Stokes equations points to its possible use for engineering estimates of the hydrodynamic characteristics of secondary flows in a thin layer on the side wall of a rotating cylinder.

NOTATION

v_x, v_y, v_z, flow velocity components; r, z, radial and axial coordinates; p, pressure; ρ, medium density; ν, kinematic viscosity; L, cylinder length; h, fluid layer thickness; ω, angular cylinder rotation velocity; Re, Reynolds number; u_0, v_0, azimuthal and radial velocity components of a medium in a flow core; δ_0, δ_1, boundary layer thicknesses on fixed and rotating cylinder ends; a, b, parameters for the velocity profiles in boundary layers; A, dimensionless parameter; ψ, stream function ($v_z = \partial\psi/\partial x$, $v_x = -\partial\psi/\partial z$).

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THE SPATIAL SCALES OF HETEROGENEITIES IN THE DIRECT-CURRENT DISCHARGE STREAM

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The results of a calculational-experimental investigation of spatial scale heterogeneities in a gas stream with direct-current discharge are presented. The numerical modeling was made within the framework of the Navier-Stokes equations with a distributed source of input energy density in a gas. The experimental study was conducted by the Talbot-interferometry method. The Shtrel number was used as the criterion of the optical homogeneity of the stream.

The creation of efficient technological electrical discharge lasers [1] is to a considerable extent connected with provision of high homogeneity of the active medium in the region where the output radiation is formed. The presence of interconnected processes during the realization in the stream of an independent glow discharge makes the modeling of the flow region in the gas discharge chamber (GDC) more complex, proving the necessity of studying and obtaining data about the stream structure [2]. The results of a calculational-experimental study of the stream region in a direct-current discharge GDC are presented. At the same time, the main attention is given to studying the structure of spacial scale heterogeneities in a gas flow and to their influence on the optical quality of the active medium.

The energy put into a self-sustaining glow discharge is divided mainly between translational and oscillatory degrees of freedom of molecules. When $E/N < (1\text{-}2)\cdot 10^{-12}$ W·m^2, up to 80% of the discharge energy [3] may be put into the oscillatory degrees of freedom of molecules. The rest of the energy goes into the translational degrees, causing quick warm-up of the gas and, thus, changing its gas dynamic parameters. This is also proved by the results of a numerical study [4]. At the same time, a low gas velocity and density (in GDC length Re ~ 9000) lead to the growth of boundary layers on side wall-electrodes, the thickness of which increases quickly along the stream under the conditions of volume energy input. A variation of gas dynamic stream parameters induces the development of refraction index gradients and a corresponding distortion of the medium's optical homogeneity.

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